Buckling Criterion for Linear Viscoelastic Columns

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THE buckling of linear viscoelastic columns has been analyzed in Refs. 1–5. Kempner and Pohle[3] and Hilton[5] demonstrated that a column made of a linear viscoelastic material does not possess a finite critical time. This corrects the result stated by Freudenthal.[1] In all of these cases the column was assumed to have an initial curvature. It is the purpose of this paper to consider the buckling of a perfectly straight, linear viscoelastic column and to apply the buckling criterion described in Refs. 6–10.

The well known equation for an initially straight elastic column is

$$EI(d^2y/dx^2) + Py = 0 \tag{1}$$

where E is the modulus of elasticity of the column material, I the moment of inertia of the column cross section, P the applied axial load, y the lateral deflection, and x the distance along the column. The equation for a linear viscoelastic column can be obtained through the use of the correspondence principle by replacing the elastic constant E by its corresponding linear viscoelastic operator $E(p)$ where p represents the time derivative $\partial/\partial t$. Thus, the equation for an initially straight linear viscoelastic column is

$$E(p)I(d^2y/dx^2) + Py = 0 \tag{2}$$

The solution of Eq. (2) can be obtained using separation of variables. A solution of the form

$$y(x,t) = X(x)\ T(t) \tag{3}$$

produces the equations

$$(d^2X/dx^2) + \alpha^2 X = 0 \tag{4}$$

$$[\alpha^2 I\ E(p) - P]T = 0 \tag{5}$$

where α^2 is the separation constant. In general, Eq. (5) has solutions of the form

$$T(t) = \sum_i A_i e^{\beta_i t} \tag{6}$$

where the β_i's are determined from the polynominal equation

$$\alpha^2 IE(\beta_i) - P = 0 \tag{7}$$

The function $T(t)$ represents the magnitude of the deflection, and thus the rate of deflection is either positive or negative depending upon the signs of the β_i's. With this in mind, two types of critical loads can be defined; a lower critical load corresponding to a zero rate of deflection and an upper critical load corresponding to an infinite rate of deflection. The lower critical load is determined by setting $\beta_i = 0$ in Eq. (7). This produces

$$\alpha^2 = P/E(0)I \tag{8}$$

which, when substituted into Eq. (4), gives

$$(d^2X/dx^2) + [P/E(0)I]X = 0 \tag{9}$$

This equation is of the same form as Eq. (1) for an elastic column. Thus, the lower critical load can now be determined from the corresponding elastic critical load by replacing the elastic constant E by $E(0)$. For example, the lower critical load for a linear viscoelastic column with pinned ends and length L would be

$$P = \pi^2 E(0)I/L^2 \tag{10}$$

Similarly, the upper critical load is determined by replacing the elastic constant E by $E(\infty)$. From this discussion, if P is the applied load, the buckling criterion can be summarized as follows: $0 < P <$ lower critical load (deflection decreases with time); $P =$ lower critical load (deflection is constant); lower critical load $< P <$ upper critical load (deflection increases with time); $P =$ upper critical load (infinite deflection immediately).

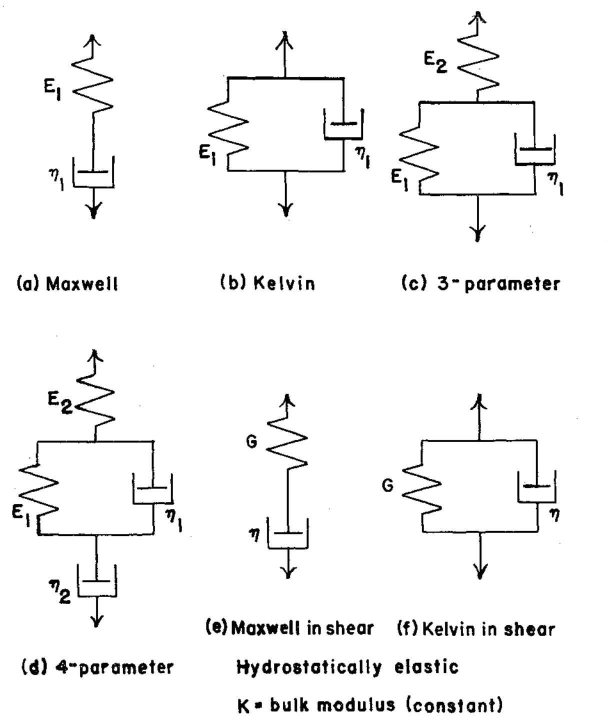

Fig. 1 Various viscoelastic models.

Several additional comments should be made at this point. First of all, if the applied load is less than the upper critical load, there is no finite critical time, which is easily ascertained by examination of Eq. (6). This verifies the results found in Refs. 3 and 5. Secondly, as pointed out by Hilton,[5] the possibility of multiple roots occuring in Eq. (7) does not alter the general results. Finally, it is noted that the amplitude coefficients A_i are left undetermined. This is to be expected since it also occurs in the elastic stability problem when no initial curvature is assumed.

Table 1 gives the lower and upper critical loads for the viscoelastic models represented in Fig. 1. It is assumed the column has pinned ends and length L.

Table 1 Buckling loads for viscoelastic models illustrated in Fig. 1

Model	Lower critical load	Upper critical load
(a)	0	$\dfrac{\pi^2 E_1 I}{L^2}$
(b)	$\dfrac{\pi^2 E_1 I}{L^2}$	∞
(c)	$\dfrac{\pi^2 E_1 E_2 I}{L^2 (E_1 + E_2)}$	$\dfrac{\pi^2 E_2 I}{L^2}$
(d)	0	$\dfrac{\pi^2 E_2 I}{L^2}$
(e)	0	$\dfrac{9\pi^2 KGI}{L^2 (3K + G)}$
(f)	$\dfrac{9\pi^2 KGI}{L^2(3K + G)}$	$\dfrac{9\pi^2 KI}{L^2}$

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A Boundary-Layer Problem Associated with Magnetogasdynamic Channel Flow

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In connection with the motion of electrically conducting fluids through electric and magnetic fields, it is useful to consider flow in a channel as a possible means of energy extraction or addition. The simplest form of analysis, based on the familiar one-dimensional approximation, excludes the very important influences of viscous stresses and heat flux. Although one may account for surface friction and heat transfer in an approximate manner as Dahlberg[1] has discussed, there is no possibility for computing the detailed effects. One way of (partially) correcting this defect is based on the idea that, for some distance downstream of the entrance, the immediate effects of the walls may be confined to thin boundary regions. The central portion of the flow is regarded as a one-dimensional problem, the solution of which provides the "freestream" conditions set in the boundary-layer problem.

Although there have been a number of discussions of boundary layers on plates when the fluid is electrically conducting, there seems to have been much less work on problems arising in channel flow. The closest to the present discussion is that by Kerrebrock[2] and Hale,[3] who treat special cases; the latter incorporates Hall currents, which are ignored here. Moffat[4] has discussed the problem and used an integral method to analyze the boundary layers on the side (insulating) walls but with the pressure constant.

The results outlined here are restricted to the necessary conditions for the existence of a class of similarity solutions to a particular set of boundary-layer equations for compressible flow. The formulation is based on certain assumptions that have been discussed in detail, for example, in Refs. 2 and 5. It is necessary only to state the conventions adopted; the electric field E, magnetic induction B, and "freestream" flow speed u are positive in the negative z, positive y, and positive x directions, respectively. The current density j is then positive in the positive z direction for a generator. It is supposed that E and B are established by means external to the flow so that the local current density in a generator is, for a scalar electrical conductivity,

$$j = \sigma(uB - E) \tag{1}$$

If the "magnetic Reynolds number" is small, then the magnetic field associated with the flow of currents in the gas is small, and it is consistent to assume B to be caused essentially by external means only. The corresponding assumption that the electric field strength E also is due to sources outside the flow implies that the gas should be electrically neutral at all points. This situation often prevails because of the large forces that arise if there is significant charge separation. However, in the present problem, somewhat closer examination is necessary. Consider the boundary layers on the electrodes of a generator; away from the regions near the side (insulating) walls, the current must be uniform in the direction normal to the surface since no current flow is permitted, within the approximations adopted here, in the axial directions. Thus, since u and σ vary through the boundary layer, Eq. (1) can be satisfied only if E varies as well. This means that there is space charge within the boundary layer, and the drop in potential across the boundary layer is different from that in an equal distance in the freestream. Equation (1), applied within the boundary layer, is really an equation determining E. The contribution of net charge density, $\epsilon_0\nabla\cdot\mathbf{E}$, to the current flow is a small correction that may be neglected.

On the other hand, the electric field can be uniform within the boundary layers on the side walls except in the regions near the electrodes. One can suppose that the tendency to charge neutrality does prevail and that Eq. (1) is an equation for j in the viscous region. Clearly, at all points for which $uB < E$ ($u \to 0$ near the surface, of course), j is locally negative, and current flows in the direction opposite to that of the current outside the boundary layer. Hence, there exists the possibility for closed-current loops within the channel. The flow in the corners, where the boundary layers on the electrodes and the side walls merge, constitutes a very much more difficult problem that has not been investigated further. It may be a practically important question because of the concomitant power losses.

The situation in an accelerator is different, for $uB < E$ everywhere; the current flows in the same direction at all points. However, since $|j| = \sigma|E - uB|$ and both u and $1/\sigma$ decrease in the region near the side walls, $|j|$ must increase. The flow adjacent to these walls, therefore, offers a "short circuit" path relative to the flow in the central region of the channel. These remarks indicate that the solutions to the boundary-layer equations for the flow over the side walls and electrodes should differ in a qualitative, as well as quantitative, respect, and indeed, one such distinction appears already in the similarity solutions enumerated below.

It is assumed that the fluid behaves as a perfect gas with constant Prandtl number and specific heats. The equations of conservation for the flow outside the boundary layers are those appropriate to a one-dimensional flow; the corresponding boundary-layer equations can be deduced from the Navier-Stokes equations written for an electrically conducting fluid and may be found, for example, in Ref. 3. Following the usual approach, one seeks conditions under which the boundary-layer partial differential equations may be reduced to nonlinear, ordinary differential equations; all

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